# Starter

## Fully expand $(3 - x)^5$

**Binomial series**

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

# D Sequences and series

## D1

Understand and use the binomial expansion of $(a + bx)^n$ for positive integer $n$; the notations $n!$ and $n$C$r$; link to binomial probabilities.

Extend to any rational $n$, including its use for approximation; be aware that the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ (Proof not required.)

### Teaching guidance

Students should be able to:

- answer questions requiring the full expansion of expressions of the form $(a + bx)^n$, where $n$ is a small positive integer
- find the coefficients of particular powers of $x$ (complete expansion not required)
- understand factorial notation.

# D Sequences and series

## D1

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Extend to any rational $n$, including its use for approximation; be aware that the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ (Proof not required.)

Notes

- The notations $\binom{n}{r}$, ${}_n\mathrm{C}_r$ and ${}^n\mathrm{C}_r$ must all be recognised. Any of these may be used.
- The $x$ in $(a+bx)^n$ may be a simple function of $x$, eg $\left(2-\frac{1}{x}\right)^4$

# back at your expansion of $(3 - x)^5$

$$
\begin{array}{lcccccc}
 & {}^5C_0 & {}^5C_1 & {}^5C_2 & {}^5C_3 & {}^5C_4 & {}^5C_5 \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(3 - x)^5 = & 1 & 5 & 10 & 10 & 5 & 1 \\
 & 3^5 & 3^4 & 3^3 & 3^2 & 3^1 & 3^0 \\
 & (-x)^0 & (-x)^1 & (-x)^2 & (-x)^3 & (-x)^4 & (-x)^5
\end{array}
$$

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$(3 - x)^5 = 243 – 405 + 270x^2 – 90x^3 + 15x^4 – x^5$

## Example 1:

Find the $x^3$ term in the expansion of $(3 – x)^5$

$(3)(-x)^5$ 3

C = $10(-x^3)(9) = -90x^3$

2

# 2.2 The Binomial Theorem

**<u>Example 2</u>:**

Find the coefficient of $x^6$ in the expansion of
$(1 - 2x)^8$

6
6

$(1)(-2x)$ 8

C

2

$= (28)\ (64x^6)\ (1)$

$= 1792x^6$

So the coefficient is 1792

# 2.2 The Binomial Theorem

## <u>Example 3</u>:

Find the coefficient of $x^3$ in the expansion of
(2 -

$(3 - 2x)^7$

$= 3^7 + \binom{7}{1}3^6(-2x) + \binom{7}{2}3^5(-2x)^2 + \binom{7}{3}3^4(-2x)^3 + \ldots$

$= 2187 - 10\,206x + 20\,412x^2 - 22\,680x^3 + \ldots$

First find the first four terms of the binomial expansion of $(3 - 2x)^7$.

$(2 + x)(2187 - 10\,206x + 20\,412x^2 - 22\,680x^3 + \ldots)$

Now expand the brackets $(2 + x)(3 - 2x)^7$.

$x^3$ term $= 2 \times (-22\,680x^3) + x \times 20\,412x^2$

$= -24\,948x^3$

There are two ways of making the $x^3$ term: (constant term × $x^3$ term) and ($x$ term × $x^2$ term).

The coefficient of $x^3$ in the binomial expansion of $(2 + x)(3 - 2x)^7$ is $-24\,948$.

# 2.2 The Binomial Theorem

## Example 4

Calculate the value of the constant if the coefficient of the  term in the expansion of  is 160.

# 2.2 The Binomial Theorem

## Example 5

$g(x) = (1 + kx)^{10}$, where $k$ is a constant.

Given that the coefficient of $x^3$ in the binomial expansion of $g(x)$ is 15, find the value of $k$.

$$\therefore k = \frac{1}{2}$$

# 2.2 The Binomial Theorem

## Example 6

A football squad consists of 13 players. Use the formula

$$nCr = \frac{n!}{r!(n-r)!}$$

to show, without a calculator, that there are 78 possible combinations of choosing a team of 11 players from this squad.

$$\frac{13!}{11!\left(13-11\right)!} = \frac{13 \times 12 \times 11 \times 10 \times \ldots}{\left(11 \times 10 \times 9 \times \ldots\right) \times 2} = \frac{13 \times 12}{2} = \frac{156}{2} = 78$$

# 2.2 The Binomial Theorem

## Example 7

In the expansion of , the coefficient of  is five times the coefficient of . Find the value of .

Coefficient of :

Coefficient of :

$$4\binom{n}{2}=5\times 2\binom{n}{1}$$

$$4\left[\frac{n(n-1)(n-2)\ldots}{2!(n-2)\ldots}\right]=10\left[\frac{n(n-1)(n-2)\ldots}{1!(n-1)\ldots}\right]$$

$$4\left[\frac{n(n-1)}{2}\right]=10n$$

# 2.2 The Binomial Theorem

## Example 7

In the expansion of , the coefficient of  is five times the coefficient of . Find the value of .

$$4\left[\frac{n(n-1)}{2}\right]=10n$$

$$2n(n-1)=10n$$

$$2n^2-12n=0$$

$$n(n-6)=0 \qquad \therefore n=6$$

# 2.2 The Binomial Theorem

## Example 8

**a** Write down the first three terms, in ascending powers of $x$, of the binomial expansion of $(1 + qx)^8$, where $q$ is a non-zero constant.

**b** Given that, in the expansion of $(1 + qx)^8$, the coefficient of $x$ is $-r$ and the coefficient of $x^2$ is $7r$, find the value of $q$ and the value of $r$.

**a** $(1 + qx)^8$

$$= 1^8 + \binom{8}{1}1^7(qx)^1 + \binom{8}{2}1^6(qx)^2 + \ldots$$

$$= 1 + 8qx + 28q^2x^2 + \ldots$$

**b** $8q = -r$ and $28q^2 = 7r$

$8q = -4q^2$

$4q^2 + 8q = 0$

$4q(q + 2) = 0$

$q = -2,\ r = 16$

**Problem-solving**

There are two unknowns in this expression. Your expansion will be in terms of $q$ and $x$.

Using $28q^2 = 7r$, $r = 4q^2$ and $-r = -4q^2$.

$q$ is non-zero so $q = -2$.